Summation Formulae for a Series Involving the I – Functions of Several Variables

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***Abstract-** In this paper, we derive two new summation formulae for a class of series involving I – functions of several variables. These functions, which generalize many known special functions, play a crucial role in various branches of Mathematical analysis and applied Mathematics. By employing suitable techniques of summation and transformation, we establish compact and elegant expressions for these series under certain convergence conditions. These results contribute to the ongoing development of summation theorems for generalized special functions and may further applications in Mathematical analysis.*

***Keywords:** I – function of several variables, Gamma function, Special functions, Pocchammer symbol*

1. Introduction

Notations and Results used:

$(a)_n$ stands for $a(a+1)\ldots(a+n-1)$.

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, n \geq 1 \qquad (1.1)$$

${}_1\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_p$ stands for $\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)};A_1\right),\left(a_2;\alpha_2^{(1)},\ldots,\alpha_2^{(r)};A_2\right),\ldots,\left(a_p;\alpha_p^{(1)},\ldots,\alpha_p^{(r)};A_p\right)$.

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan, and Santha Kumari [4, p.38] is defined and represented as:

$$I[z_1,\ldots\ldots,z_r]$$

$$= I^{0,n:\, m_1,n_1;\ldots;m_r,n_r}_{p,q:\, p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j)_Q : {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ldots;{}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\,\omega)^r}\int_{L_1}\ldots\int_{Lr}\theta_1(s_1)\ldots\theta_r(s_r)\ \phi\,(s_1,\ldots,s_r)\ z_1^{s_1}\ldots z_r^{s_r}\,ds_1\ldots ds_r, \qquad (1.2)$$

where $\phi\,(s_1,\ldots,s_r)$ and $\theta_i(s_i),\ i=1,2,\ldots,r$ are given by,

$$\phi(s_1,\dots,s_r)=\frac{\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=1}^{q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\prod_{j=n+1}^{p}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)},\tag{1.3}$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}.\tag{1.4}$$

Also $z_i\neq 0\ (i=1,\dots,r)$, $\omega=\sqrt{-1}$, $m_j,n_j,p_j,q_j\ (j=1,\dots,r)$,n ,p ,q are non-negative integers such that $0\le n\le p$, $q\ge 0$, $0\le m_j\le q_j$, $0\le n_j\le p_j$ $(j=1,2\dots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}\ (j=1,2,\dots,p\ ,\ i=1,2\dots,r)$, $\beta_j^{(i)}\ (j=1,2,\dots,q,\ \ i=1,2,\dots,r)$, $\gamma_j^{(i)}\ (j=1,2,\dots,p_i,\ i=1,2,\dots,r)$, and $\delta_j^{(i)}\ (j=1,2,\dots,q_i,\ i=1,2,\dots,r)$ are positive numbers. $a_j\ (j=1,2,\dots,p)$, $b_j\ (j=1,2,\dots,q)$, $c_j^{(i)}\ (j=1,2,\dots,p_i,\ i=1,2,\dots,r)$, and $d_j^{(i)}\ (j=1,2,\dots,q_i,\ i=1,2,\dots,r)$ are complex numbers. The exponents $A_j\ (j=1,2,\dots,p)$, $B_j\ (j=1,2,\dots,q)$, $C_j^{(i)}\ (j=1,2,\dots,p_i,\ i=1,2,\dots,r)$, and $D_j^{(i)}\ (j=1,2,\dots,q_i,\ i=1,2,\dots,r)$ of various gamma functions may take non integer values. The I-function of r-variables is analytic if

$$\Psi_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\ \le 0, i=1,2,\dots,r.$$

The integral (1.2) converges absolutely if $\left|\arg(z_i)\right|<\frac{1}{2}\Delta_i\pi\ ,\ i=1,2,\dots,r$ where

$$\Delta_i=-\sum_{j=n+1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)}$$

$$+\sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{i}>0\ .\tag{1.5}$$

For further details refer [4].

Dixon's theorem [2]

$${}_3F_2\left[a,b,c;\,1+a-b,1+a-c;1\right]=\frac{\Gamma\left(1+a/2\right)\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)\Gamma\left(1+a/2-b-c\right)}{\Gamma\left(1+a\right)\Gamma\left(1+a/2-b\right)\Gamma\left(1+a/2-c\right)\Gamma\left(1+a-b-c\right)},\tag{1.6}$$

provided $\mathrm{Re}(a-2b-2c)>-2$

$$ {}_3F_2\left[a,b,c;\,d,e;x\right]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{x^n}{n!}. \tag{1.7}$$

Sharma B.L, [7]

$$ \mathrm{F}_{2:0;0}^{2:1;1}\left[\begin{matrix}\alpha, \quad \beta & :T;\delta; \\ 2\alpha,\frac{1}{2}(1+\beta+t+\delta):\_;\_; & \end{matrix}1,1\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}(T)_m(\delta)_n}{(2\alpha)_{m+n}\left(\frac{1+\beta+t+\delta}{2}\right)_{m+n}m!n!}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}T+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\beta-\frac{1}{2}T-\frac{1}{2}\delta\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}T+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{1}{2}T-\frac{1}{2}\delta\right)} \tag{1.8}$$

provided $\mathrm{Re}(2\alpha-\beta-T-\delta)>-1$.

2. Result - I: A summation formula for a series involving *I*-functions of several variables

$$\sum_{k=0}^{\infty}\frac{(b)_k(c)_k}{k!}I_{p,q:p_1+3,q_1+3;\,\dots;p_r,q_r}^{0,n:\;m_1,n_1+3\;\;;\,\dots\,;m_r,n_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_1\\ \\ I_2\end{matrix}\right]=I_{p,q:p_1+4,q_1+4;\,\dots;p_r,q_r}^{0,n:\;m_1,n_1+4\;\;;\,\dots\,;m_r,n_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_3\\ \\ I_4\end{matrix}\right], \tag{2.1}$$

where

$$I_1={}_1\left(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)};A_j\right)_p:(1-a-k,\lambda;1),(-a+b,\lambda;1),(-a+c,\lambda;1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1};\dots;{}_1\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r},$$

$$I_2={}_1\left(b_j;\beta_j^{(1)},\dots,\beta_j^{(r)};B_j\right)_q:{}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(1-a,\lambda;1),(-a+b-k,\lambda;1),(-a+c-k,\lambda;1);{}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2};\dots;{}_1\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r},$$

$$I_3={}_1\left(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)};A_j\right)_p:(-\tfrac{a}{2},\tfrac{\lambda}{2};1),(-a+b,\lambda;1),(-a+c,\lambda;1),(-\tfrac{a}{2}+b+c,\tfrac{\lambda}{2};1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1};\dots;{}_1\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r},$$

$$I_4={}_1\left(b_j;\beta_j^{(1)},\dots,\beta_j^{(r)};B_j\right)_q:{}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(-a,\lambda;1),(-\tfrac{a}{2}+b,\tfrac{\lambda}{2};1),(-\tfrac{a}{2}+c,\tfrac{\lambda}{2};1),(-a+b+c,\lambda;1);{}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2};\dots;{}_1\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r}.$$

provided

(i) $\lambda>0$,

(ii) $\mathrm{Re}\left(2+a-2b-2c+\lambda\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$, $j=1,2,\dots,m_1$, $i=1,2,\dots,r$,

(iii) $|\arg(z_i)|<\frac{1}{2}\Delta_i\pi, i=1,2,\dots,r$,

where Δ_i is given by (1.5).

Proof:

Express the *I*-function of *r*- variables on the left-hand side of (2.1) as a contour integral using (1.2), then simplify it by successively applying (1.7) and (1.6), leading to a reduced form. Finally, by interpreting this resulting integral with the help of (1.2), we arrive the right-hand side of (2.1).

3. Special Cases:

When $r=2$, (2.1) reduces to:

$$\sum_{k=0}^{\infty}\frac{(b)_k(c)_k}{k!}I^{0,n:\ m_1,n_1+3\ ;\ m_2,n_2}_{p,q:p_1+3,q_1+3;\ p_2,q_2}\left[\begin{array}{c|c} z_1 & I_1 \\ z_2 & I_2 \end{array}\right]=I^{0,n:\ m_1,n_1+4\ ;\ m_2,n_2}_{p,q:p_1+4,q_1+4;\ p_2,q_2}\left[\begin{array}{c|c} z_1 & I_3 \\ z_2 & I_4 \end{array}\right] \tag{3.1}$$

where

$$I_1={}_1\left(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j\right)_p:(1-a-k,\lambda;1),(-a+b,\lambda;1),(-a+c,\lambda;1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1};{}_1\left(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)}\right)_{p_2},$$

$$I_2={}_1\left(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j\right)_q:{}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(1-a,\lambda;1),(-a+b-k,\lambda;1),(-a+c-k,\lambda;1);{}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2},$$

$$I_3={}_1\left(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j\right)_p:(-\tfrac{a}{2},\tfrac{\lambda}{2};1),(-a+b,\lambda;1),(-a+c,\lambda;1),(-\tfrac{a}{2}+b+c,\tfrac{\lambda}{2};1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1};{}_1\left(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)}\right)_{p_2},$$

$$I_4={}_1\left(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j\right)_q:{}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(-a,\lambda;1),(-\tfrac{a}{2}+b,\tfrac{\lambda}{2};1),(-\tfrac{a}{2}+c,\tfrac{\lambda}{2};1),(-a+b+c,\lambda;1);{}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2}.$$

provided that the conditions are similar to that of (2.1) with $r=2$.

Put $n=p=q=0$, $r=1$, and specializing the parameters in (2.1), (2.1) reduces to:

$$\sum_{k=0}^{\infty}\frac{(b)_k(c)_k}{k!}I^{m\ \ n+3}_{p+3,q+3}\left[z_1\left|\begin{array}{c} I_1 \\ I_2 \end{array}\right.\right]=I^{m\ \ n+4}_{p+4,q+4}\left[z_1\left|\begin{array}{c} I_3 \\ I_4 \end{array}\right.\right] \tag{3.2}$$

where

$$I_1=(1-a-k,\lambda;1),(-a+b,\lambda;1),(-a+c,\lambda;1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_p,$$

$$I_2={}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_q,(1-a,\lambda;1),(-a+b-k,\lambda;1),(-a+c-k,\lambda;1),$$

$$I_3=(-\tfrac{a}{2},\tfrac{\lambda}{2};1),(-a+b,\lambda;1),(-a+c,\lambda;1),(-\tfrac{a}{2}+b+c,\tfrac{\lambda}{2};1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_p,$$

$$I_4={}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_q,(-a,\lambda;1),(-\tfrac{a}{2}+b,\tfrac{\lambda}{2};1),(-\tfrac{a}{2}+c,\tfrac{\lambda}{2};1),(-a+b+c,\lambda;1).$$

provided that the conditions are similar to that of (2.1) with $n=p=q=0$ and $r=1$.

4. Result - II: A Summation formula for a double series involving *I*-function of several variables

$$\sum_{\omega_1=0}^{\infty}\sum_{\omega_2=0}^{\infty}\frac{(\beta)_{\omega_1+\omega_2}(T)_{\omega_1}(\delta)_{\omega_2}}{\left(\frac{1+\beta+T+\delta}{2}\right)_{\omega_1+\omega_2}\omega_1!\omega_2!}I^{0,n:\ m_1,n_1+2\ ;\ldots;m_r,n_r}_{p,q:p_1+2,q_1+2;\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_1\\ \\ I_2\end{matrix}\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\beta}{2}+\frac{T}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)\Gamma\left(\frac{1}{2}+\frac{T}{2}+\frac{\delta}{2}\right)}I^{0,n:\ m_1,n_1+2\ ;\ldots;m_r,n_r}_{p,q:p_1+2,q_1+2;\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_3\\ \\ I_4\end{matrix}\right] \quad (4.1)$$

where,

$$I_1={}_1\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_p:(1-\alpha-\omega_1-\omega_2,\lambda;1),(1-2\alpha,2\lambda;1),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1};\ldots;{}_1\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r},$$

$$I_2={}_1\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_q:{}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(1-\alpha,\lambda;1),(1-2\alpha-\omega_1-\omega_2,2\lambda;1);{}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2};\ldots;{}_1\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r},$$

$$I_3={}_1\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j\right)_p:\left(\tfrac{1}{2}-\alpha,\lambda;1\right),\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2}+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda;1\right),{}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1};\ldots;{}_1\left(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)}\right)_{p_r},$$

$$I_4={}_1\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_q:{}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2},\lambda;1\right),\left(\tfrac{1}{2}-\alpha+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda;1\right);{}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2};\ldots;{}_1\left(d_j^{(r)},\delta_j^{(r)};D_j^{(r)}\right)_{q_r}.$$

provided

(i) $\lambda>0$,

(ii) $\mathrm{Re}\left(2\alpha-\beta-T-\delta+2\lambda\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$, $j=1,2,\ldots,m_1$, $i=1,2,\ldots,r$,

(iii) $|\arg(z_i)|<\frac{1}{2}\Delta_i\pi$, $i=1,2,\ldots,r$,

where Δ_i is given by (1.5) .

5. Special Cases:

When $r=2$, (4.1) becomes:

$$\sum_{\omega_1=0}^{\infty}\sum_{\omega_2=0}^{\infty}\frac{(\beta)_{\omega_1+\omega_2}(T)_{\omega_1}(\delta)_{\omega_2}}{\left(\frac{1+\beta+T+\delta}{2}\right)_{\omega_1+\omega_2}\omega_1!\omega_2!}I^{0,n:\ m_1,n_1+2\ ;m_2,n_2}_{p,q:p_1+2,q_1+2;p_2,q_2}\left[\begin{matrix}z_1\\ z_2\end{matrix}\middle|\begin{matrix}I_1\\ I_2\end{matrix}\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\beta}{2}+\frac{T}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)\Gamma\left(\frac{1}{2}+\frac{T}{2}+\frac{\delta}{2}\right)}I^{0,n:\ m_1,n_1+2\ ;m_2,n_2}_{p,q:p_1+2,q_1+2;p_2,q_2}\left[\begin{matrix}z_1\\ z_2\end{matrix}\middle|\begin{matrix}I_3\\ I_4\end{matrix}\right], \quad (5.1)$$

where

$$I_1 = {}_1\left(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j\right)_p : (1-\alpha-\omega_1-\omega_2,\lambda;1),(1-2\alpha,2\lambda;1), {}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1}; {}_1\left(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)}\right)_{p_2},$$

$$I_2 = {}_1\left(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j\right)_q : {}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(1-\alpha,\lambda;1),(1-2\alpha-\omega_1-\omega_2,2\lambda;1); {}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2},$$

$$I_3 = {}_1\left(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j\right)_p : \left(\tfrac{1}{2}-\alpha,\lambda;1\right),\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2}+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda;1\right), {}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1}; {}_1\left(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)}\right)_{p_2},$$

$$I_4 = {}_1\left(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j\right)_q : {}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2},\lambda;1\right),\left(\tfrac{1}{2}-\alpha+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda;1\right); {}_1\left(d_j^{(2)},\delta_j^{(2)};D_j^{(2)}\right)_{q_2}.$$

provided that the conditions are similar to that of (4.1) with $r = 2$.

When $n = p = q = 0$ and $r = 1$ and specializing the parameters of (5.1), (5.1) reduces to:

$$\sum_{\omega_1=0}^{\infty}\sum_{\omega_2=0}^{\infty}\frac{(\beta)_{\omega_1+\omega_2}(T)_{\omega_1}(\delta)_{\omega_2}}{\left(\frac{1+\beta+T+\delta}{2}\right)_{\omega_1+\omega_2}\omega_1!\,\omega_2!} I_{p+2,q+2}^{m,\,n+2}\left[z\middle|\begin{matrix}I_1\\ I_2\end{matrix}\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\beta}{2}+\frac{T}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)\Gamma\left(\frac{1}{2}+\frac{T}{2}+\frac{\delta}{2}\right)} I_{p+2,q+2}^{m,\,n+2}\left[z\middle|\begin{matrix}I_3\\ I_4\end{matrix}\right] \tag{5.2}$$

where

$$I_1 = (1-\alpha-\omega_1-\omega_2,\lambda;1),(1-2\alpha,2\lambda;1), {}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1},$$

$$I_2 = {}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},(1-\alpha,\lambda;1),(1-2\alpha-\omega_1-\omega_2,2\lambda;1),$$

$$I_3 = \left(\tfrac{1}{2}-\alpha,\lambda;1\right),\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2}+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda;1\right), {}_1\left(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)}\right)_{p_1},$$

$$I_4 = {}_1\left(d_j^{(1)},\delta_j^{(1)};D_j^{(1)}\right)_{q_1},\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2},\lambda;1\right),\left(\tfrac{1}{2}-\alpha+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda;1\right).$$

provided that the conditions are similar to that of (4.1) with $n = p = q = 0$ and $r = 1$.

When $C_j^{(1)} = D_j^{(1)} = 1$, (5.2) reduces to the result involving the H-functions of one variable.

$$\sum_{\omega_1=0}^{\infty}\sum_{\omega_2=0}^{\infty}\frac{(\beta)_{\omega_1+\omega_2}(T)_{\omega_1}(\delta)_{\omega_2}}{\left(\frac{1+\beta+T+\delta}{2}\right)_{\omega_1+\omega_2}\omega_1!\,\omega_2!} H_{p+2,q+2}^{m,\,n+2}\left[z\middle|\begin{matrix}I_1\\ I_2\end{matrix}\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\beta}{2}+\frac{T}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)\Gamma\left(\frac{1}{2}+\frac{T}{2}+\frac{\delta}{2}\right)} H_{p+2,q+2}^{m,\,n+2}\left[z\middle|\begin{matrix}I_3\\ I_4\end{matrix}\right]$$

where

$$I_1 = (1-\alpha-\omega_1-\omega_2,\lambda),(1-2\alpha,2\lambda),\ _1\left(c_j^{(1)},\gamma_j^{(1)}\right)_{p_1},$$

$$I_2 = {}_1\left(d_j^{(1)},\delta_j^{(1)}\right)_{q_1},(1-\alpha,\lambda),(1-2\alpha-\omega_1-\omega_2,2\lambda),$$

$$I_3 = \left(\tfrac{1}{2}-\alpha,\lambda\right),\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2}+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda\right),\ _1\left(c_j^{(1)},\gamma_j^{(1)}\right)_{p_1},$$

$$I_4 = {}_1\left(d_j^{(1)},\delta_j^{(1)}\right)_{q_1},\left(\tfrac{1}{2}-\alpha+\tfrac{\beta}{2},\lambda\right),\left(\tfrac{1}{2}-\alpha+\tfrac{T}{2}+\tfrac{\delta}{2},\lambda\right).$$

provided that the conditions are similar to that of (4.1) with $n=p=q=0$, $C_j^{(1)}=D_j^{(1)}=1$ and $r=1$.

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