Applications Of I-Functions of Several Variables on Deriving the Exact Distribution of Products and Ratios of Generalized Gamma Random Variables

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***Abstract-** In this paper a density function is derived for the product and ratios of generalized Gamma variables.*
***Keywords:** Multivariable I–functions, The density function, Melin Transform, Mathematical expectation*

Introduction

The I- functions of r-variables was introduced by Prathima and Vasudevan Nambisan [8, p.38] and is defined and represented as,

$$I[z_1,\ldots\ldots,z_r]$$

$$= I^{0,N:\, m_1,n_1;\ldots;m_r,n_r}_{P,Q:\, p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j)_P:{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j)_Q\,:{}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ldots;{}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\ \omega\ )^r}\int_{L_1}\ldots\int_{Lr}\theta_1(s_1)\ldots\theta_r(s_r\ )\ \phi\ (s_1,\ldots,s_r)\ \ z_1^{s_1}\ldots z_r^{s_r}\,ds_1\ldots ds_r, \tag{1.1}$$

where,

$\varphi\ (s_1,\ldots,s_r)\, and\ \theta_i(s_i),\ \ i=1,2,\ldots,r\ \ are\ given\, by$

$$\phi\ (s_1,\ldots,s_r) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=1}^{Q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\ \prod_{j=N+1}^{P}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}, \tag{1.2}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}}(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}}(1 - c_j^{(i)} + \gamma_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}}(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}}(c_j^{(i)} - \gamma_j^{(i)} s_i)}, \tag{1.3}$$

Also $z_i \neq 0$ $(i = 1,...,r)$, $\omega = \sqrt{-1}$, m_j, n_j, p_j, q_j $(j = 1,...,r)$, N , P , Q are non-negative integers such that $0 \le N \le P$, $Q \ge 0$, $0 \le m_j \le q_j$, $0 \le n_j \le p_j$ $(j = 1,2....,r)$ (not all zero simultaneously). $\alpha_j^{(i)}(j = 1,2,...,P, i = 1,2...,r)$, $\beta_j^{(i)}(j = 1,2,...,Q, i = 1,2,...,r)$, $\gamma_j^{(i)}(j = 1,2,..., p_i, i = 1,2,...,r)$, and $\delta_j^{(i)}(j = 1,2,...,q_i, i = 1,2,...,r)$ are positive numbers. $a_j(j = 1,2,...,P)$, $b_j(j = 1,2,...,Q)$, $c_j^{(i)}(j = 1,2,..., p_i, i = 1,2,...,r)$, and $d_j^{(i)}(j = 1,2,...,q_i, i = 1,2,...,r)$ are complex numbers. The exponents $A_j(j = 1,2,...,P)$, $B_j(j = 1,2,...,Q)$, $C_j^{(i)}(j = 1,2,..., p_i, i = 1,2,...,r)$, and $D_j^{(i)}(j = 1,2,...,q_i, i = 1,2,...,r)$ of various gamma functions may take non integer values.

The I-function of r-variables is analytic if

$$\Psi_i = \sum_{j=1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \le 0, i = 1,2,...,r.$$

The integral (1.1) converges absolutely if,

$$|\arg(z_i)| < \frac{1}{2}\Delta_i \pi,\ i = 1,2,...,r$$

Where,

$$\Delta_i = \left(-\sum_{j=n+1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} \delta_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \gamma_j^{(i)} \right) > 0 \tag{1.4}$$

Mathai and Saxena [3]

The probability density function $f(x)$ of random variable X is defined as

$$\int_{-\infty}^{+\infty} f(x)dx = 1,\ f(x) \ge 0,\ \forall x \tag{1.5}$$

A random variable X is called a generalized Gamma variant if it has the density function

$$f(x) = \frac{\beta a^{\frac{\alpha}{\beta}}}{\Gamma\left(\frac{\alpha}{\beta}\right)} x^{\alpha-1} e^{-ax\beta}, x > 0, a > 0, \alpha > 0, \beta > 0$$
$$= 0,\ elsewhere \tag{1.6}$$

Let $X_1, X_2, ..., X_k$ be a simple random variable independently and identically distributed according to the probability law (1.6).

Let $$Y = \frac{X_1, X_2, ..., X_m}{X_{m+1}, X_{m+2}, ..., X_k} \quad (1.7)$$

If the density $g(y)$ of Y exists, then $(s-1)^{th}$ moment of Y about the origin is given by

$$E\left(Y^{s-1}\right) = \prod_{j=1}^{m}\left[E\left(X_j^{s-1}\right)\right]\prod_{j=m+1}^{k}\left[E\left(X_j^{-(s-1)}\right)\right]$$
$$= \frac{a^{\frac{m-n}{\beta}}\Gamma^m\left(\frac{\alpha-1}{\beta}+\frac{s}{\beta}\right)\Gamma^{k-m}\left(\frac{\alpha+1}{\beta}-\frac{s}{\beta}\right)}{\Gamma^k\left(\frac{\alpha}{\beta}\right)a^{\frac{(m-n)s}{\beta}}} \quad (1.8)$$

Where $n = k - m$ and E denote the mathematical Expectation.

2. Main Results

If $X_{1k}, X_{2k}, ..., X_{pk}, k = 1, 2, ...r$ are independent random variable having density function,

$$f(x_k) = \frac{\beta_k a_k^{\frac{\alpha_k}{\beta_k}} x_k^{\alpha_k - 1} e^{-a_k x_k \beta_k}}{\Gamma\left(\frac{\alpha_k}{\beta_k}\right)}, x_k > 0, a_k > 0, \alpha_k > 0, \beta_k > 0, k = 1, 2, ..., r \quad (2.1)$$

And if $$Y_k = \frac{X_{1k}, X_{2k}, ..., X_{mk}}{X_{(m+1)k}, X_{(m+2)k}, ..., X_{pk}}, k = 1, 2, ...r \quad (2.2)$$

Then the density $g(y_1, y_2, ..., y_r)$ of $(Y_1, Y_2, ..., Y_r)$ whenever it exist is given by

$$g(y_1, y_2, ..., y_r) = cI^{0,0:\ 1,1;...;1,1}_{0,0:\ 1,1;...;1,1}\left[\begin{array}{c|c} a^{\frac{m_1-n_1}{\beta_1}}y_1 & I_1 \\ \vdots & \\ a^{\frac{m_r-n_r}{\beta_r}}y_r & I_2 \end{array}\right], \quad 0 < y_k < \infty,\ k = 1, 2, ..., r \quad , (2.3)$$

Where,

$$I_1 = \left(1 - \frac{(\alpha_1+1)}{\beta_1}, \frac{1}{\beta_1}; n_1\right); ...; \left(1 - \frac{(\alpha_r+1)}{\beta_r}, \frac{1}{\beta_r}; n_r\right)$$

$$I_2 = \left(\frac{(\alpha_1-1)}{\beta_1}, \frac{1}{\beta_1}; m_1\right); ...; \left(\frac{(\alpha_r-1)}{\beta_r}, \frac{1}{\beta_r}; m_r\right)$$

Here $$c=\frac{a_1^{\frac{m_1-n_1}{\beta_1}}\dots a_r^{\frac{m_r-n_r}{\beta_r}}}{\Gamma^{\beta_1}\left(\frac{\alpha_1}{\beta_1}\right)\dots\Gamma^{\beta_r}\left(\frac{\alpha_r}{\beta_r}\right)} \tag{2.4}$$

Provided (i) *I*-functions of several variable exist and

(ii) $\frac{\alpha_k-1}{\beta_k}+\upsilon_k\neq-\frac{\alpha_k+1}{\beta_k}-\mu_k>0,\ \upsilon_k,\mu_k=0,1,2,\dots\ \&\ k=1,2,\dots r$

Proof:

If the density $g(y_k)$ of Y_k exists, then $(s_k-1)^{th}$ moment of Y_k about the origin is given by,

$$E\left(Y_k^{s_k-1}\right)=\prod_{j=1}^{m_k}\left[E\left(X_{j_k}^{s_k-1}\right)\right]\prod_{j=m_k+1}^{p_k}\left[E\left(X_{j_k}^{-(s_k-1)}\right)\right],\ k=1,2,\dots,r \tag{2.5}$$

That is,

$$E\left(Y_1^{s_1-1},\dots,Y_r^{s_r-1}\right)=\frac{a_1^{\frac{m_1-n_1}{\beta_1}}\Gamma^{m_1}\left(\frac{\alpha_1-1}{\beta_1}+\frac{s_1}{\beta_1}\right)\Gamma^{p_1-m_1}\left(\frac{\alpha_1+1}{\beta_1}-\frac{s_1}{\beta_1}\right)\dots a_r^{\frac{m_r-n_r}{\beta_r}}\Gamma^{m_r}\left(\frac{\alpha_r-1}{\beta_r}+\frac{s_r}{\beta_r}\right)\Gamma^{p_r-m_r}\left(\frac{\alpha_r+1}{\beta_r}-\frac{s_r}{\beta_r}\right)}{\Gamma^{p_1}\left(\frac{\alpha_1}{\beta_1}\right)\dots\Gamma^{p_r}\left(\frac{\alpha_r}{\beta_r}\right)a_1^{\left(\frac{m_1-n_1}{\beta_1}\right)s_1}\dots a_1^{\left(\frac{m_1-n_1}{\beta_1}\right)s_r}} \tag{2.6$$

Where, $n_k=p_k-m_k,k=1,2,\dots,r$, using (1.8) and, E denote the Mathematical Expectation.

The density function $g\left(y_1,y_2,\dots,y_r\right)$ is obtained by taking inverse Melin transform of (2.3)

$$g\left(y_1,y_2,\dots,y_r\right)=\frac{\prod_{k=1}^{r}a_k^{\frac{m_k-n_k}{\beta_k}}}{\Gamma^{p_k}\left(\frac{\alpha_k}{\beta_k}\right)}\int_{c_1-\infty}^{c_1+\infty}\dots\int_{c_r-\infty}^{c_r+\infty}\prod_{j=1}^{r}\Gamma^{m_k}\left(\frac{\alpha_k-1}{\beta_k}+\frac{s_k}{\beta_k}\right)\Gamma^{m_k}\left(\frac{\alpha_k+1}{\beta_k}-\frac{s_k}{\beta_k}\right)a_k^{\frac{-(m_k-n_k)s_k}{\beta_k}}y^{-s_k}ds_1\dots ds_r \tag{2.7}$$

Now interpreting (2.7) with the help of (1.1), (2.3) is obtained.

3. Special cases

If $n=p=q=0$ (2.3) becomes

$$g\left(y_1,y_2,\dots,y_r\right)=cI_{1,1}^{1,1}\left[a_1^{\frac{m_1-n_1}{\beta_1}y_1}\middle|\begin{matrix}\left(1-\frac{(\alpha_1+1)}{\beta_1},\frac{1}{\beta_1};n_1\right)\\ \left(\frac{(\alpha_1-1)}{\beta_1},\frac{1}{\beta_1};m_1\right)\end{matrix}\right]\dots I_{1,1}^{1,1}\left[a_r^{\frac{a_r-m_r}{\beta_r}y_r}\middle|\begin{matrix}\left(1-\frac{(\alpha_r+1)}{\beta_r},\frac{1}{\beta_r};n_r\right)\\ \left(\frac{(\alpha_r-1)}{\beta_r},\frac{1}{\beta_r};m_r\right)\end{matrix}\right], \tag{3.1}$$

Provided the conditions similar to that of (2.3)

Transforming (2.3) into H-Functions of several variables, (2.3) becomes,

$$g\left(y_1,y_2,...,y_r\right)=cH^{0,0:\ m_1,n_1;...;m_r,n_r}_{0,0:\ n_1,m_1;...;m_r,n_r}\left[\begin{matrix} a^{\frac{m_1-n_1}{\beta_1}y_1} \\ \vdots \\ a^{\frac{m_r-n_r}{\beta_r}y_r}\end{matrix}\middle|\begin{matrix} ---;C_1,C_2,...,C_r \\ \\ ---;D_1,D_2,...,D_r\end{matrix}\right],$$

$$C_i=\left(1-\frac{(\alpha_i+1)}{\beta_i},\frac{1}{\beta_i}\right);...;\left(1-\frac{(\alpha_i+1)}{\beta_i},\frac{1}{\beta_r}\right),\ i=1,2,...,r$$

$$D_i=\left(\frac{(\alpha_i-1)}{\beta_i},\frac{1}{\beta_i}\right);...;\left(\frac{(\alpha_i+1)}{\beta_i},\frac{1}{\beta_r}\right),\ i=1,2,...,r$$

$$g\left(y_1,y_2,...,y_r\right)=cH^{m_1,n_1}_{n_1,m_1}\left[a_1^{\frac{m_1-n_1}{\beta_1}y_1}\middle|\begin{matrix}\left(1-\frac{(\alpha_1+1)}{\beta_1},\frac{1}{\beta_1}\right)\\ \left(\frac{(\alpha_1-1)}{\beta_1},\frac{1}{\beta_1}\right)\end{matrix}\right]...\ H^{m_r,n_r}_{n_r,m_r}\left[a_r^{\frac{a_r-m_r}{\beta_r}y_r}\middle|\begin{matrix}\left(1-\frac{(\alpha_r+1)}{\beta_r},\frac{1}{\beta_r}\right)\\ \left(\frac{(\alpha_r-1)}{\beta_r},\frac{1}{\beta_r}\right)\end{matrix}\right],$$

When $r=2$, (2.3) reduces to

$$g\left(y_1,y_2\right)=cI^{0,0:\ 1,1;1,1}_{0,0:\ 1,1;1,1}\left[\begin{matrix}a_1^{\frac{m_1-n_1}{\beta_1}y_1}\\ a_2^{\frac{m_2-n_2}{\beta_2}y_2}\end{matrix}\middle|\begin{matrix}\left(1-\frac{(\alpha_1+1)}{\beta_1},\frac{1}{\beta_1};m_1\right);\left(1-\frac{(\alpha_2+1)}{\beta_2},\frac{1}{\beta_2};m_2\right)\\ \left(\frac{(\alpha_1-1)}{\beta_1},\frac{1}{\beta_1};m_1\right);\left(\frac{(\alpha_2-1)}{\beta_2},\frac{1}{\beta_2};m_2\right)\end{matrix}\right]$$

$$g\left(y_1,y_2\right)=cI^{1,1}_{1,1}\left[a_1^{\frac{m_1-n_1}{\beta_1}y_1}\middle|\begin{matrix}\left(1-\frac{(\alpha_1+1)}{\beta_1},\frac{1}{\beta_1};m_1\right)\\ \left(\frac{(\alpha_1-1)}{\beta_1},\frac{1}{\beta_1};m_1\right)\end{matrix}\right]I^{1,1}_{1,1}\left[a_2^{\frac{m_2-n_2}{\beta_2}y_2}\middle|\begin{matrix}\left(1-\frac{(\alpha_2+1)}{\beta_2},\frac{1}{\beta_2};m_2\right)\\ \left(\frac{(\alpha_2-1)}{\beta_2},\frac{1}{\beta_2};m_2\right)\end{matrix}\right], \quad (3.2)$$

Where $c = \dfrac{a_1^{\frac{m_1-n_1}{\beta_1}} a_r^{\frac{m_2-n_2}{\beta_2}}}{\Gamma\left(\dfrac{\alpha_1}{\beta_1}\right)\Gamma\left(\dfrac{\alpha_2}{\beta_2}\right)}$,

provided the conditions similar to that of (2.3)

When $r = 1$ and specializing the parameters, (2.3) reduces to the result given by Mathai [3,p83]

4. Conclusions: In this work**,** a density function is derived for the product and ratios of generalized Gamma variables.

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